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L^p Inequalities for Polynomials

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1. INTRODUCTION

1.1. Let $\mathscr{T}_n$ be the class of all trigonometric polynomials

$$t_n(\theta) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu\theta}$$

of degree n. It was found by Zygmund [15] that if $t_n \in \mathscr{T}_n$, then, for $1 \leqslant p < +\infty$,

$$\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |t_n'(\theta)|^p \, d\theta\right)^{1/p} \leqslant n \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |t_n(\theta)|^p \, d\theta\right)^{1/p} =: \|t_n\|_{[-\pi,\pi],p}. \quad (1)$$

Since (1) was deduced from M. Riesz's interpolation formula [12] by means of Minkowski's inequality, it was not clear whether the restriction on p was indeed essential. This question was open for a long time. Finally, Arestov [2] proved that (1) remains true for $0 < p < 1$ and indeed for $p = 0$ as well, where

$$\|t_n\|_{[-\pi,\pi],0} := \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log |t_n(\theta)| \, d\theta\right).$$

The difficulty which was associated with Zygmund's inequality (1) is characteristic of several other L^p inequalities involving polynomials,

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trigonometric polynomials, etc. The purpose of this paper is to mention a few which, like (1), can be extended to $p \in [0, 1)$.

1.2. Let $\mathscr{P}_n$ be the set of all polynomials

$$P(z) = \sum_{v=0}^{n} a_v z^v$$

of degree at most n. For $P \in \mathscr{P}_n$ define

$$\|P\|_p := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |P(e^{i\theta})|^p \, d\theta\right)^{1/p} \qquad (0 < p < +\infty),$$

$$\|P\|_\infty := \max_{|z|=1} |P(z)|$$

and

$$\|P\|_0 := \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |P(e^{i\theta})| \, d\theta\right).$$

The above result of Zygmund and Arestov says that, in particular, for $P \in \mathscr{P}_n$ and $0 \leqslant p \leqslant +\infty$,

$$\|P'\|_p \leqslant n \, \|P\|_p, \tag{2}$$

where equality holds if and only if $P(z) = cz^n$. If $P(z) \neq 0$ for $|z| < 1$ then, for $1 \leqslant p \leqslant +\infty$, inequality (2) can be replaced by [4, 7]

$$\|P'\|_p \leqslant n \, \|P\|_p / \|1 + z^n\|_p. \tag{3}$$

We prove

THEOREM 1. *Let $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < 1$. Then* (3) *holds for all* $p \in [0, +\infty]$.

1.3. It is a simple consequence of a classical result of Hardy [6] that if $P \in \mathscr{P}_n$, then [11, Theorem 5.5], for $R \geqslant 1$ and $p \geqslant 0$,

$$\|P(Rz)\|_p \leqslant R^n \, \|P\|_p, \tag{4}$$

where equality is attained if and only if $P(z) = cz^n$. If $P(z) \neq 0$ for $|z| < 1$, then for $1 \leqslant p \leqslant +\infty$ inequality (4) can be replaced by [1, 3]

$$\|P(Rz)\|_p \leqslant \frac{\|1 + R^n z^n\|_p}{\|1 + z^n\|_p} \cdot \|P\|_p. \tag{5}$$

We prove

THEOREM 2. *Let $P \in \mathscr{P}_n$ and $P(z) \neq 0$ for $|z| < 1$. Then* (5) *holds for all* $p \in [0, +\infty]$.

1.4. If

$$P(z) = \sum_{v=0}^{n} a_v z^v \in \mathscr{P}_n$$

and a_u, a_v $(u < v)$ are two coefficients such that for no other coefficient $a_w \neq 0$ we have $w \equiv u \bmod (v - u)$, then [14, 5, 9] for every $p \in [1, +\infty]$,

$$|a_u| + |a_v| \leqslant 2 \, \|P\|_p / \|1 + z^n\|_p. \tag{6}$$

This result is best possible. We prove

THEOREM 3. *In the case $u = 0$, $v = n$, inequality* (6) *holds for all* $p \in [0, +\infty]$.

Remark. It may be mentioned that inequality (6) in its full generality does not extend to $p \in [0, 1)$. To see this let

$$P(z) = (1 + z)^4 = 1 + 4z + 6z^2 + 4z^3 + z^4.$$

Then the pair of indices $(u, v) = (1, 3)$ is clearly admissible. But $|a_1| + |a_3| = 8$, whereas for $p = \frac{1}{2}$,

$$2 \, \|P\|_p / \|1 + z^n\|_p = \frac{8}{\|1 + z\|_{1/2}} < \frac{8}{\|1 + z\|_0} = 8.$$

As an application of Theorem 3 we mention.

COROLLARY. *Consider a polynomial $\prod_{v=1}^{n} (z - \zeta_v)$. Then, for $1 \leqslant k \leqslant n$ and all $p \in [0, +\infty]$,*

$$|\zeta_1 \zeta_2 \cdots \zeta_{k-1}| + |\zeta_k \zeta_{k+1} \cdots \zeta_n| \leqslant 2 \, \|P\|_p / \|1 + z^n\|_p. \tag{7}$$

This result extends Theorem 2 in [10] and lends itself to the kind of applications mentioned therein.

2. A LEMMA

For $\gamma = (\gamma_0, ..., \gamma_n) \in \mathbb{C}^{n+1}$ and

$$P(z) = \sum_{v=0}^{n} a_v z^v,$$

we define

$$\Lambda_\gamma P(z) = \sum_{\nu=0}^{n} \gamma_\nu a_\nu z^\nu.$$

The operator Λ_γ is said to be *admissible* if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C}: |z| \leqslant 1\}$,

(ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C}: |z| \geqslant 1\}$.

LEMMA [2, Theorem 4]. *Let $\phi(x) = \psi(\log x)$, where ψ is a convex non-decreasing function on $\mathbb{R}$. Then for all $P \in \mathscr{P}_n$ and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \phi(|\Lambda_\gamma P(e^{i\theta})|)\, d\theta \leqslant \int_0^{2\pi} \phi(c(\gamma, n)\, |P(e^{i\theta})|)\, d\theta, \tag{8}$$

where $c(\gamma, n) = \max(|\gamma_0|, |\gamma_n|)$.

In particular, the lemma applies with $\phi: x \mapsto x^p$ for every $p \in (0, +\infty)$ and with $\phi: x \mapsto \log x$ as well. Therefore we have

$$\|\Lambda_\gamma P\|_p \leqslant c(\gamma, n)\, \|P\|_p \qquad (0 \leqslant p < +\infty). \tag{9}$$

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. According to a theorem of Laguerre as stated in [4], if $P \in \mathscr{P}_n$ does not vanish in $K := \{z \in \mathbb{C}: |z| < 1\}$, then

$$nP(z) - (z - \zeta)\, P'(z) \neq 0 \qquad \text{for} \quad z \in K \text{ and } \zeta \in K.$$

Setting $\zeta = -e^{-i\alpha} z$, we readily see that the operator Λ defined by

$$\Lambda P(z) := (e^{i\alpha} + 1)\, zP'(z) - ne^{i\alpha} P(z)$$

is admissible and so by (9)

$$\int_0^{2\pi} \left| (e^{i\alpha} + 1) \frac{d}{d\theta} P(e^{i\theta}) - ine^{i\alpha} P(e^{i\theta}) \right|^p d\theta \leqslant n^p \int_0^{2\pi} |P(e^{i\theta})|^p\, d\theta \tag{10}$$

for $p > 0$. Rearranging the left-hand side of (10) and integrating the inequality with respect to α on $[0, 2\pi]$, we obtain

$$\int_0^{2\pi} \int_0^{2\pi} \left| \frac{d}{d\theta} P(e^{i\theta}) \right|^p \cdot |1 + e^{i\alpha} R(\theta)|^p\, d\theta\, d\alpha \leqslant 2\pi n^p\, \|P\|_p^p, \tag{11}$$

where

$$R(\theta) := \left(\frac{d}{d\theta} P(e^{i\theta}) - inP(e^{i\theta})\right)\Big/\frac{d}{d\theta} P(e^{i\theta}).$$

It is known [4, Theorem 2] that if $P(z) \neq 0$ for $|z| < 1$, then $|R(\theta)| \geqslant 1$, and therefore by a theorem of Hardy

$$\int_0^{2\pi} |1 + e^{i\alpha} R(\theta)|^p \, d\alpha \geqslant \int_0^{2\pi} |1 + e^{i\alpha}|^p \, d\alpha$$

for all $\theta \in [0, 2\pi]$. Using this in (11), the desired result follows immediately for $p > 0$. The extension to $p = 0$ is obtained by continuity.

Proof of Theorem 2. For $R \geqslant 1$ and $\gamma \in \mathbb{R}$, the polynomial

$$\sum_{\nu=0}^{n} \binom{n}{\nu} (R^\nu + e^{i\gamma} R^{n-\nu})\, z^\nu$$

has all its zeros on the unit circle (see [8, Problem 26, p. 108]). Hence, if

$$P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu \in \mathscr{P}_n$$

does not vanish for $|z| < 1$, then by Szegö's convolution theorem [13] the same is true for

$$\begin{aligned} \Lambda P(z) &:= (1 + e^{i\gamma} R^n)\, a_0 + (R + e^{i\gamma} R^{n-1})\, a_1 z + \cdots + (R^n + e^{i\gamma})\, a_n z^n \\ &= P(Rz) + e^{i\gamma} R^n P(z/R). \end{aligned}$$

Therefore Λ is an admissible operator. Applying (9) we obtain

$$\int_0^{2\pi} |P(Re^{i\theta}) + e^{i\gamma} R^n P(e^{i\theta}/R)|^p \, d\theta \leqslant |R^n e^{i\gamma} + 1| \cdot \|P\|_p^p \qquad (12)$$

for $p > 0$. Since

$$f(z) := z^n \overline{P(1/\bar{z})}/P(z)$$

is holomorphic for $|z| \leqslant 1$ with $|f(z)| = 1$ on the unit circle, it follows from the maximum principle that $|f((1/R)\, e^{i\theta})| \leqslant 1$ for $1/R < 1$ and so

$$|R^n P(e^{i\theta}/R)/P(Re^{i\theta})| \geqslant 1 \qquad (R \geqslant 1). \qquad (13)$$

Now, integrating (12) with respect to γ on $[0, 2\pi]$ and using (13), the desired result is obtained in the same way as Theorem 1.

Proof of Theorem 3. The operator Λ defined by

$$\Lambda\left(\sum_{\nu=0}^{n} a_\nu z^\nu\right) := a_0 + a_n z^n$$

is obviously admissible. Hence by (9)

$$\int_0^{2\pi} |a_0 + a_n e^{in\theta}|^p \, d\theta \leqslant \int_0^{2\pi} \left|\sum_{\nu=0}^{n} a_\nu e^{i\nu\theta}\right|^p d\theta \tag{14}$$

for all $p > 0$. From the inequality

$$\left|\frac{1 + re^{i\theta}}{1 + e^{i\theta}}\right| \geqslant \frac{1+r}{2}$$

we deduce that

$$|a_0| + |a_n| \leqslant 2\left(\int_0^{2\pi} |a_0 + a_n e^{i\theta}|^p \, d\theta \Big/ \int_0^{2\pi} |1 + e^{i\theta}|^p \, d\theta\right)^{1/p}.$$

Using this in conjunction with (14), the desired result follows.

The corollary is obtained by applying Theorem 3 to the polynomial

$$Q(z) := P(z) \prod_{j=k}^{n} \left(\frac{\bar{\zeta}_j z - 1}{z - \zeta_j}\right).$$

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